1. Let $\mathbb{R}^*$ denote the group of nonzero real numbers under multiplication. Let $\mathbb{R}^+$ denote the group of positive real numbers under multiplication. Show that $\mathbb{R}^*$ is the internal direct product of $\mathbb{R}^+$ and the subgroup $\{+1, -1\}$.

2. In $\mathbb{Z}$, let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $\mathbb{Z} = H + K$. Is $\mathbb{Z}$ the internal direct product of $H$ and $K$?

3. If $\varphi$ is a homomorphism from $G$ to $H$ and $\sigma$ is a homomorphism from $H$ to $K$, show that $\sigma \varphi$ is a homomorphism from $G$ to $K$. How are $\ker \varphi$ and $\ker \sigma \varphi$ related? If $\varphi$ and $\sigma$ are onto and $G$ is finite, describe $[\ker \sigma \varphi : \ker \varphi]$ in terms of $|H|$ and $|K|$.

4. Prove that $(A \oplus B)/(A \oplus \{e\}) \approx B$. (Note that since $A \oplus \{e\} \approx A$, this gives a concrete way of understanding how external direct products and factor groups behave in a way similar to multiplication and division of numbers.)

5. Suppose that $k$ is a divisor of $n$. Prove that $\mathbb{Z}_n/\langle k \rangle \approx \mathbb{Z}_k$. (Hint: Use the first isomorphism theorem. You will need to take care to confirm that your map is a homomorphism.)

6. Suppose that $\varphi$ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and $\ker \varphi = \{0, 10, 20\}$. If $\varphi(23) = 9$, determine all elements that map to 9.

7. Suppose that there is a homomorphism $\varphi$ from $\mathbb{Z}_{17}$ to a group $G$ and that $\varphi$ is not one-to-one. Determine $\varphi$.

8. How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$? How many homomorphisms are there from $\mathbb{Z}_{20}$ to (but not necessarily onto) $\mathbb{Z}_8$?

9. If $\varphi$ is a homomorphism from $\mathbb{Z}_{30}$ onto a group of order 5, determine the kernel of $\varphi$.

10. Prove that the mapping $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ given by $(a, b) \mapsto a - b$ is a homomorphism. What is the kernel of $\varphi$? Describe the set $\varphi^{-1}(3)$.

11. If $K$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$, prove that $K/(K \cap N)$ is isomorphic to $KN/N$. (This is the Second Isomorphism Theorem.)