1. Show that \( \gcd(a, bc) = 1 \) if and only if \( \gcd(a, b) = 1 \) and \( \gcd(a, c) = 1 \).

2. If \( a \) and \( b \) are integers and \( n \) is a positive integer, prove that \( a \mod n = b \mod n \) if and only if \( n \) divides \( a - b \).

3. Let \( a \) and \( b \) be integers and \( d = \gcd(a, b) \). If \( a = da' \) and \( b = db' \), show that \( \gcd(a', b') = 1 \).

4. Let \( n \) be a fixed positive integer greater than 1. If \( a \mod n = a' \) and \( b \mod n = b' \), prove that \( (a + b) \mod n = (a' + b') \mod n \) and \( (ab) \mod n = (a'b') \mod n \).

5. Let \( a \) and \( b \) be positive integers and let \( d = \gcd(a, b) \) and \( m = \lcm(a, b) \). If \( t \) divides both \( a \) and \( b \), prove that \( t \) divides \( d \). If \( s \) is a multiple of both \( a \) and \( b \), prove that \( s \) is a multiple of \( m \).

6. Let \( n \) and \( a \) be positive integers and \( d = \gcd(a, n) \). Show that the equation \( ax \mod n = 1 \) has a solution if and only if \( d = 1 \). (To say that \( ax \mod n = 1 \) has a solution means that there is at least one integer \( s \) such that \( as \mod n = 1 \). In other words, think of \( x \) as the variable here.)

7. For every positive integer \( n \), use induction to prove that a set with exactly \( n \) elements has exactly \( 2^n \) subsets (counting the empty set and the entire set).

8. Let \( \mathbb{Z} \) be the set of integers. If \( a, b \in \mathbb{Z} \), define \( a \sim b \) if \( ab \geq 0 \). Is \( \sim \) an equivalence relation on \( \mathbb{Z} \)? If so, prove that it is. If not, explain why not.

9. Let \( \mathbb{Z} \) be the set of integers. If \( a, b \in \mathbb{Z} \), define \( a \sim b \) if \( a + b \) is even. Prove that \( \sim \) is an equivalence relation and determine the equivalence classes of \( \mathbb{Z} \).

10. Suppose that \( f : X \to Y \) and \( g : Y \to Z \). Show that if \( f \) and \( g \) are both one-to-one, then so is the composition \( gf \). Show that if \( f \) and \( g \) are both onto, then so is the composition \( gf \).

Note that Problems 1, 2, and 6 are “if and only if” problems. To show that (statement A) if and only if (statement B), you need to show that (statement A) implies (statement B) AND that (statement B) implies (statement A).