Notation for integrals of scalar fields

In general, we denote the integral of a scalar field $f$ over any shape $S$ by $\int_S f \, dA$; many other context-dependent notations are used for this operation in the text and elsewhere:

...over an interval $I$: $\int_a^b f(t) \, dt$, where $I = [a,b]$.

...over a curve $C$: Path integral; if $C$ is parametrized via a path $c$:
$$\int_C f(x,y,z) \, ds$$
(often $\oint_C \ldots$ if the curve is closed).

...over a surface $S$: $\iint_S f(x,y,z) \, dS$
(often $\iiint_S \ldots$ if the surface is closed).

...over a region $R$ in the plane: $\iint_R f(x,y) \, dA$.

...over a region $R$ in space: $\iiint_R f(x,y,z) \, dV$.

Notation for integrals of vector fields

- A line integral is the integral $\int_C \vec{F} \cdot \vec{T} \, ds$ of a vector field $\vec{F}$ along a curve $C$ in direction $\vec{T}$, interpreted as circulation of a velocity field or work done by a force field.

If $C$ is parametrized via a path $c$:
$$\int_C \vec{F} \cdot \vec{T} \, ds, \quad \int_C \vec{F} \cdot \vec{T} \, ds, \quad \int_C \vec{F} \cdot d\vec{r}, \quad \int_c P \, dx + Q \, dy + R \, dz$$
(often $\oint_C \ldots$ if the curve is closed).

- The integral $\int_S \vec{F} \cdot \vec{n} \, dS$ of a vector field $\vec{F}$ through a surface $S$ in direction $\vec{n}$, interpreted as the flux of the vector field through the surface ($\vec{n}$ giving the “positive” flux direction).

$$\iint_S \vec{F} \cdot \vec{n} \, dS, \quad \iint_S \vec{F} \cdot \vec{n} \, dS$$
(often $\iiint_S \ldots$ if the surface is closed).
Notation for coordinates, mappings, and magnification

Any integral computed in the text is computed either via the FTC, by slicing and Fubini’s Theorem, or by parametrization and pull-back. Particularly when “coordinate change” is used in the text, it is not always evident from the equations or notation, but the basic principle of parametrization and pull-back is at their core:

\[
\int_S f = \int_R (f \circ \Phi) \cdot \mathcal{M}
\]
if S is parametrized via \( \Phi : R \to S \).

Some examples from the text of “integration formulae” for parametrization and pull-back in different situations are listed below:

• General regions in the plane:
  \[
  \int \int_D f(x,y) \, dA = \int \int_{D'} f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv \quad \text{(p. 409).}
  \]

• General regions in space:
  \[
  \int \int \int_D f(x,y,z) \, dV = \int \int \int_{D'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, du \, dv \, dw \quad \text{(p. 421).}
  \]

• Polar regions:
  \[
  \int \int_D f(x,y) \, dA = \int \int_{D'} f(r \cos \theta, r \sin \theta) \cdot r \, dr \, d\theta \quad \text{(p. 406).}
  \]

• Cylindrical regions:
  \[
  \int \int \int_W f(x,y,z) \, dV = \int \int \int_{W'} f(r \cos \theta, r \sin \theta, z) \cdot r \, dz \, dr \, d\theta \quad \text{(p. 423).}
  \]

• Spherical regions:
  \[
  \int \int \int_W f(x,y,z) \, dV = \int \int \int_{W'} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \quad \text{(p. 424).}
  \]

• Path integrals:
  \[
  \int_c f \, ds = \int_a^b \frac{f(c(t)) \| c'(t) \|}{\mathcal{M} = \| Dc \|} \, dt \quad \text{(p. 319).}
  \]

• Line integrals:
  \[
  \int_c \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(c(t)) \cdot c'(t) \, dt \quad \left( \frac{\mathcal{M}}{\mathcal{M}} \right. \text{ cancels} \quad \text{(p. 327).}
  \]

• Surface integrals:
  \[
  \int_S f \, dS = \int \int_D f(r(u,v)) \| \vec{N}(u,v) \| \, dA \quad \text{(p. 464).}
  \]
  \[
  \int_S \vec{F} \cdot d\vec{S} = \int \int_D \vec{F}(r(u,v)) \cdot \vec{N}(u,v) \, dA \quad \text{(p. 476).}
  \]