The metric topology of $\mathbb{E}^n$ (and $\mathbb{R}^n$)

We can compute the distance between two points $p, q \in \mathbb{E}^n$ as $\|p - q\|$; this metric (way of measuring distance) lets us view $\mathbb{E}^n$ as a metric space, because the following three properties hold:

1. $\|p - q\| \geq 0$, with equality only if $p = q$  
   \textit{positive-definiteness}

2. $\|p - q\| = \|q - p\|$  
   \textit{symmetry}

3. $\|p - r\| \leq \|p - q\| + \|q - r\|$  
   \textit{the triangle inequality}

In a metric space allows us to define what it means for a set to be “open,” – a concept that helps us to more cleanly state and prove facts about shapes and continuous functions (which are the focus of the mathematical field of topology):

Given $x \in \mathbb{E}^n$ and $r > 0$, the \textbf{open ball} with center $x$ and radius $r$ is the set $D_r(x) = \{x' \in \mathbb{E}^n : \|x' - x\| < r\}$.

A subset $U \subset \mathbb{E}^n$ is called called a \textbf{neighborhood of $x$} if there exists an open ball around $x$ that is fully contained in $U$, i.e., $x$ is entirely surrounded by points of $U$, rather than being at its boundary. Formally:

$\exists r > 0$ for which $D_r(x) \subset U.$

A set $U \subset \mathbb{E}^n$ that is a neighborhood of every one of its points is called \textbf{open}. Formally, $U$ is open if:

$\forall x \in U \exists r > 0$ for which $D_r(x) \subset U.$

To show that a subset $U \subset \mathbb{E}^n$ is open:

$\forall x \in U \exists r > 0$

1. Given an arbitrary $x \in U$, determine a positive value $r$ that you claim will work.

2. For such $x$ and $r$, prove that: If $\|x' - x\| < r$, then $x' \in U$. 

A point $x$ in an open set $U$
Techniques of proof

A few ways to prove that \( P \Rightarrow Q \) (i.e., “If \( P \), then \( Q \)”):

- **Direct**  
  Assume \( P \) as a hypothesis, and use this to show that \( Q \) is true.

- **Contrapositive**  
  Prove instead that \((\neg Q) \Rightarrow (\neg P)\).

- **Contradiction**  
  Prove that \((P \text{ and } \neg Q)\) is impossible.

- **Definition**  
  Prove that \((Q \text{ or } \neg P)\) is a true statement.

Except in very specific circumstances, the logic of an argument flows in one direction only; so to prove logical equivalence (“if and only if”), be prepared to make one argument for each direction. In other words, to prove that \( P \Leftrightarrow Q \), you could:

- Connect \( P \) to \( Q \) very carefully by a chain of logical equivalences. [not always possible]

  - or -  
  Prove separately that \( P \Rightarrow Q \) and that \( Q \Rightarrow P \).

Basic set theory definitions and properties

Suppose that \( A \) and \( B \) are sets, and that \( V \) is a collection of sets. Then:

- **Subsets**  
  \( A \subset B \) means \( x \in A \Rightarrow x \in B \)

- **Equality**  
  \( A = B \) means \( x \in A \iff x \in B \)  
  [or, equivalently, \( A \subset B \) and \( B \subset A \)]

- **Union**  
  \( A \cup B = \{ x : x \in A \text{ or } x \in B \} \)  
  \( \bigcup V = \{ x : \exists V \in V \text{ with } x \in V \} \)

- **Intersection**  
  \( A \cap B = \{ x : x \in A \text{ and } x \in B \} \)  
  \( \bigcap V = \{ x : x \in V \forall V \in V \} \)

- **Difference**  
  \( A - B = \{ x : x \in A \text{ and } x \not\in B \} \)

- **Cartesian product**  
  \( A \times B = \{(a, b) : a \in A \text{ and } b \in B \} \)

- **DeMorgan’s laws**  
  \( X - (A \cup B) = (X - A) \cap (X - B) \)  
  \( X - \bigcup_{V \in V} V = \bigcap_{V \in V} (X - V) \)

  \( X - (A \cap B) = (X - A) \cup (X - B) \)  
  \( X - \bigcap_{V \in V} V = \bigcup_{V \in V} (X - V) \)

- **Emptiness**  
  \( A \neq \emptyset \iff \exists a \in A \)

  \( A \) and \( B \) are called **disjoint** if \( A \cap B = \emptyset \)

  \( V \) is called a **disjoint collection** if \( A, B \in V \Rightarrow A = B \) or \( A \cap B = \emptyset \)