The Divergence Theorem

Suppose that we have a smooth vector field $\vec{F}$ on a region $R$. Then the net expansion of $\vec{F}$ on $R$ must be balanced by net flux of $\vec{F}$ through the boundary of $R$:

$$\int_R \text{div} \vec{F} = \int_{\partial R} \vec{F} \cdot \vec{n}$$

($\vec{n}$ pointing out of $R$)

Note that:
- The vector field $\vec{F}$ must be defined and smooth on all of $R$.
- We must consider the entire boundary of $R$.
- Any flux whose normal does not point out of $R$ must be negated!

- We can also use this theorem to determine relationships between the flux of a vector field through different surfaces, by first finding a region relating them.

- If $\vec{F}$ is incompressible, then the left-hand side of the above equation will be zero, which allows us to conclude that certain fluxes are zero or equal.

Stokes’ Theorem

Suppose that we have a smooth vector field $\vec{F}$ on an orientable surface $S$. Then the net rotation of $\vec{F}$ on $S$ matches the circulation of $\vec{F}$ along the boundary of $S$:

$$\int_S (\text{curl } \vec{F}) \cdot \vec{n} = \int_{\partial S} \vec{F} \cdot \vec{T}$$

($\vec{n}$ and $\vec{T}$ compatible*)

Note that:
- The vector field $\vec{F}$ must be defined and smooth on all of $S$.
- We must consider the entire boundary of $S$.
- Circulations in which $\vec{T}$ is not compatible with $\vec{n}$ must be negated!
  * Right-handed rotation about $\vec{n}$ must move in the direction of $\vec{T}$.

- We can also use this theorem to determine relationships between the circulation of a vector field along different paths, by first finding a surface relating them.

- If $\vec{F}$ is irrotational, then the left-hand side of the above equation will be zero, which allows us to conclude that certain circulations are zero or equal.
Orientation of surfaces

To orient a smooth surface $S$ is to make a consistent choice of unit normal vector at each point of $S$. The choice of unit normal is called an orientation of $S$; we call a surface $S$ along with an orientation an oriented surface.

- A surface for which this choice is possible is called orientable.
- A surface for which this choice is impossible is called non-orientable;
  
  e.g., the Möbius band:

Curves and paths

- A curve $C$ is called closed if it has no boundary, i.e., no ends.
- An oriented curve is a curve $C$ along with a consistent choice of direction on $C$; note that this amounts to making a consistent choice of unit tangent vector at each point of $C$.

Conservative vector fields

Suppose that we have a smooth vector field $\vec{F}$ on some path-connected region $R$.

- A potential function for $\vec{F}$ is a scalar field $f$ with $\nabla f = \vec{F}$.
- FTC for gradient fields: If $f$ is a potential function for $\vec{F}$ and $\gamma$ is a path from $a$ to $b$, then $\int_{\gamma} \vec{F} \cdot \vec{T} = f(b) - f(a)$.

- $\vec{F}$ is called conservative when any one of the three following equivalent conditions holds:
  - there exists a potential function $f$ for $\vec{F}$
  - $\int_{C} \vec{F} \cdot \vec{T} = 0$ for every closed curve $C$
  - $\int_{\gamma_1} \vec{F} \cdot \vec{T} = \int_{\gamma_2} \vec{F} \cdot \vec{T}$ for any two paths $\gamma_1, \gamma_2$ with matching [start and] endpoints

Theorem: Any irrotational vector field $\vec{F}$ on a simply connected region $R$ is conservative.

Proof: Use Stokes’ Theorem and the simple-connectivity of $R$ to show that $\int_{C} \vec{F} \cdot \vec{T} = 0$ for every closed curve $C$. 