Functions and the derivative in Cartesian coordinates

Suppose that $F : \mathbb{E}^n \to \mathbb{E}^m$ is a smooth function (similarly for $F : \mathbb{E}^n \to \mathbb{R}^m$, etc.).

- Taking coordinates on the domain and range, we can represent the function $F$ via $m$ real-valued functions of $n$ variables each. For example, suppose that $F : \mathbb{E}^2 \to \mathbb{E}^3$. Taking coordinates $(u, v)$ on the domain and $(x, y, z)$ on the range, each point in the domain corresponds to some pair of coordinates $(u, v)$, and each point in the range corresponds to some triple of coordinates $(x, y, z)$. The function $F$, then, assigns to each pair of domain coordinates three range coordinates, given by coordinate functions $x(u, v), y(u, v), z(u, v)$.

- Once a function is written in terms of coordinates, we can differentiate it with respect to one coordinate at a time (as in single-variable calculus), treating the other coordinates as constants. This operation is called partial differentiation; partial differentiation with respect to the variable $v$ is denoted by $\frac{\partial}{\partial v}$. Higher-order partial derivatives can be computed iteratively, and for smooth functions, the order of differentiation does not matter. For example:

$$\frac{\partial}{\partial x}[5xy + \sin(x^3 + y)] = 5y + 3x^2 \cos(x^3 + y) \quad \text{and} \quad \frac{\partial}{\partial y}[5xy + \sin(x^3 + y)] = 5x + \cos(x^3 + y),$$

so

$$\frac{\partial}{\partial y \partial x}[5xy + \sin(x^3 + y)] = 5 - 3x^2 \sin(x^3 + y) = \frac{\partial}{\partial x \partial y}[5xy + \sin(x^3 + y)].$$

- If $p \in \mathbb{E}^n$, then $DF_p$ is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$, and thus can be represented in coordinates by an $m \times n$ matrix.

What are its entries? From linear algebra, we know that the entry in the $i^{th}$ row and $j^{th}$ column is the coefficient of the $i^{th}$ basis vector in the range when $DF_p$ is evaluated on the $j^{th}$ basis vector in the domain. But this is just the single-variable derivative of the $i^{th}$ coordinate function in the range with respect to the $j^{th}$ coordinate in the domain—i.e., a partial derivative! Thus, if $F : \mathbb{E}^n \to \mathbb{E}^m$ and we take coordinates $(x_1, \ldots, x_n)$ in the domain and $(y_1, \ldots, y_n)$ in the range, then $DF$ is given in coordinates by the matrix

$$DF = \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n}
\end{bmatrix}$$

If $p \in \mathbb{E}^n$, then we obtain a matrix for $DF_p$ by evaluating each entry at $p$. 


Fields and their derivatives in Cartesian coordinates

Suppose that \( f : \mathbb{E}^3 \to \mathbb{R} \) is a smooth scalar field and \( \vec{F} : \mathbb{E}^3 \to \mathbb{R}^3 \) is a smooth vector field.

- Using Cartesian coordinates, we can express the scalar field \( f \) simply as a function of three variables, e.g., \( f(x, y, z) = x^2 + y - z \); each \((x, y, z)\) corresponds to a point, and this expression tells us the scalar value to assign to that point in terms of its coordinates.

- The vector field \( \vec{F} \) can similarly be written in terms of three scalar fields \( F_1, F_2, F_3 \) as \( \vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \), or \( \{F_1, F_2, F_3\} \).
  For example, \( \vec{F}(x, y, z) = y \hat{i} + (\sin x) \hat{j} + x^2 \hat{k} \) tells us that at the point \( p = (\pi, 2, 3) \), \( \vec{F} \) gives the vector \( 2\hat{i} + 0\hat{j} + 3\pi^2\hat{k} = \{2, 0, 3\pi^2\} \).

- The Laplacian of \( f \) is a the scalar field \( \Delta f \) is defined by \( \Delta f = \text{div}(\text{grad} f) \); if \( \Delta f \equiv 0 \), \( f \) is called harmonic.

- The operator “\text{del}”, \( \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \) allows us to easily compute gradient, divergence, and curl in terms of partial derivatives:

  **Gradient:** \( \text{grad} f \) is the vector field \( \nabla f = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \)

  **Divergence:** \( \text{div} \vec{F} \) is the scalar field \( \nabla \cdot \vec{F} = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \)

  **Curl:** \( \text{curl} \vec{F} \) is the vector field \( \nabla \times \vec{F} = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \)

  \[ = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \]

  **Laplacian:** \( \Delta f \) is the scalar field \( \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \)

- In coordinates, the fact that \( \text{curl}(\nabla f) \equiv \vec{0} \) for a smooth scalar field \( f \) is a consequence of equality of mixed partial derivatives 
  \( \frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x} \), etc.\)